Darboux's transformation method for generating solutions of the evolution equation for Kelvin-Helmholtz waves near direct resonance

1987 J. Phys. A: Math. Gen. 20 3293

(http://iopscience.iop.org/0305-4470/20/11/032)

J. Phys. A: Math. Gen. 20 (1987) 3293-3300. Printed in the UK

Darboux's transformation method for generating solutions of the evolution equation for Kelvin–Helmholtz waves near direct resonance

Yi-shen Li†

International Centre for Theoretical Physics, Miramare, Trieste, Italy

Received 22 August 1986, in final form 10 November 1986

Abstract. In this paper, several new solutions of the evolution equation for Kelvin-Helmholtz waves near direct resonance are obtained by Darboux transformation.

1. Introduction

Ma (1984) derived an evolution equation for Kelvin-Helmholtz waves near direct resonance and the Lax pair associated with this equation was also found. Ma discussed the inverse scattering problem in the $(1+1)$-dimensional case and gave the soliton solutions of this equation but a simple solution, which does not decay to zero as $x \to \infty$, cannot be included in this method.

Recently, a method has been suggested for generating the solutions of the non-linear evolution equations (NEE) which possess a Lax pair. This method depends on the old theorem proved by Darboux. It is well known that the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0 \tag{1.1}$$

can be deduced from the compatibility condition of the Lax pair

$$\varphi_{xx} = (\lambda - u)\varphi \tag{1.2}$$

$$\varphi_t = -u_x\varphi + (2u + 4\lambda)\varphi_x. \tag{1.3}$$

Darboux had proved that if $\varphi = \varphi(x, t, \lambda)$ is the general solution of (1.2), $f = f(x, t, \lambda_0)$ is a special solution of (1.2) with $\lambda = \lambda_0$, then, by the transformation

$$\bar{\varphi} = \varphi_x - (f_x/f)\varphi \tag{1.4}$$

$$\bar{u} = u - 2(\ln f)_{xx} \tag{1.5}$$

$\bar{\varphi}$ satisfies the equation

$$\bar{\varphi}_{xx} = (\lambda - \bar{u})\bar{\varphi}. \tag{1.6}$$

It implies that (1.2) is 'invariant' under the action $\varphi \to \bar{\varphi}$, $u \to \bar{u}$. We call (1.4) and (1.5) the Darboux transformation (DT).

† On leave of absence from Department of Mathematics, University of Science and Technology of China, Hefei, China.

0305-4470/87/113293+08$02.50

The relation (1.5) has been used to deduce the x part of the Bäcklund transformation for the KdV equation by Wadati *et al* (1975): the connection between u and $\bar{u}$ is a non-linear equation.

We follow Darboux's idea in another way. We find that if φ is also a solution of (1.3) and f is also a special solution of equation (1.3) with $\lambda=\lambda_0$, then under (DT) (1.4) and (1.5), (1.3) is also 'invariant', i.e.

$$\bar{\varphi}_t=-\bar{u}_x\bar{\varphi}+(2\bar{u}+4\lambda)\bar{\varphi}_x. \tag{1.7}$$

It implies that if u is a solution of the KdV equation (1.1) then $\bar{u}$ defined by (1.5) is a new solution of the KdV equation. Moreover $\bar{\varphi}$ defined by (1.4) is a solution of (1.6) and (1.7). For another special value $\lambda=\lambda_1$, and a special solution $\bar{f}$ or $\bar{f}=\bar{\varphi}(x, t, \lambda_1)$ we obtain another new solution $\bar{\bar{u}}=\bar{u}-(\ln\bar{f})_{xx}$ for the KdV equation. This procedure can be continued.

Based on the DT (1.4) and (1.5), we can generate a series of solutions for the KdV equation by solving two linear problems without solving any tedious non-linear equations and to generate the exact solution of the NEE without the knowledge of any boundary conditions, such that we can obtain not only the soliton solution but also another type of solution. We have successfully used this method to generate the solutions for several NEE (see Li 1986a, b, 1987, Li and Gu 1987, Li and Wang 1985).

In this paper, we use this method for generating the solutions of the evolution equation for Kelvin-Helmholtz waves near direct resonance. The paper is organised as follows: in § 2, we deduce a more general non-linear evolution equation from the Lax pair and contains the equation of Kelvin-Helmholtz waves near direct resonance as a special case. In § 3, we discuss the Darboux transformation method for this equation. In § 4, several types of solutions are given.

2. A non-linear evolution equation and its Lax pair

We consider the following two linear problems:

$$\bar{D}\varphi=M\varphi \qquad M=\sum_{j=0}^{2}\begin{pmatrix}a_j & b_j\\ c_j & -a_j\end{pmatrix}\xi^j \qquad \varphi=\begin{pmatrix}\varphi_{11} & \varphi_{12}\\ \varphi_{21} & \varphi_{22}\end{pmatrix} \tag{2.1}$$

$$\varphi_t=N\varphi \qquad N=\xi N_1+N_0 \tag{2.2}$$

where

$$\bar{D}=\frac{\partial}{\partial x}+\beta\frac{\partial}{\partial y} \qquad a_2=-2\mathrm{i} \qquad b_2=c_2=0 \qquad a_1=-\mathrm{i}\alpha \qquad b_1=2q$$

$$c_1=2r \qquad a_0=-\mathrm{i}qr \qquad b_0=\mathrm{i}q_t+\alpha q \qquad c_0=-\mathrm{i}r_t+\alpha r \tag{2.3}$$

$$N_1=\begin{pmatrix}-\mathrm{i} & 0\\ 0 & \mathrm{i}\end{pmatrix} \qquad N=\begin{pmatrix}0 & q\\ r & 0\end{pmatrix}.$$

ξ is an eigenparameter, α and β are real constants and q and r are functions of x, y and t.

From $\bar{D}\partial_t\varphi=\partial_t\bar{D}\varphi$, it yields

$$M_t-\bar{D}N+MN-NM=0. \tag{2.4}$$

Substituting (2.1)-(2.3) into (2.4), we obtain

$$\begin{aligned} &\mathrm{i}q_{tt}+\alpha q_t-(q_x+\beta q_y)-2\mathrm{i}q^2 r=0 \\ &-\mathrm{i}r_{tt}+\alpha r_t-(r_x+\beta r_y)+2\mathrm{i}qr^2=0. \end{aligned} \tag{2.5}$$

This is a (1+2)-dimensional non-linear evolution equation. If $q=-r^*=S$, then equation (2.5) reduces to the equation

$$S_{tt}-\mathrm{i}\alpha S_t+\mathrm{i}(S_x+\beta S_y)+2|S|^2S=0 \tag{2.6}$$

which is equation (14) of Ma (1984).

3. Darboux transformation

We deduce the Darboux transformation of (2.2). We assume a gauge transformation

$$\bar\varphi=T\varphi \qquad T=\begin{pmatrix}-2\mathrm{i} & 0\\ 0 & 2\mathrm{i}\end{pmatrix}\xi+\begin{pmatrix}a & b\\ c & d\end{pmatrix} \qquad \bar\varphi=\begin{pmatrix}\bar\varphi_{11} & \bar\varphi_{12}\\ \bar\varphi_{21} & \bar\varphi_{22}\end{pmatrix} \tag{3.1}$$

(where a, b, c and d are functions of x, y and t). It maps the equation (2.2) into the following:

$$\bar\varphi_t=\bar N\bar\varphi \qquad \bar N=\begin{pmatrix}-\mathrm{i} & 0\\ 0 & \mathrm{i}\end{pmatrix}\xi+\begin{pmatrix}0 & \bar q\\ \bar r & 0\end{pmatrix}. \tag{3.2}$$

Then T satisfies the relation $T_t=\bar N T-TN$, i.e. a, b, c, d; q, r, $\bar q$ and $\bar r$ satisfy the relations

$$b=\bar q+q \qquad c=\bar r+r \tag{3.3a}$$

$$a_t=\bar q c-rb \qquad d_t=\bar r b-qc \tag{3.3b}$$

$$b_t=\bar q d-qa \qquad c_t=\bar r a-rd. \tag{3.3c}$$

Obviously, the polynomial of $\det T(\xi)=4\xi^2+2\mathrm{i}(a-d)+ad-bc$ has two zeros ξ_1 and ξ_2 such that $\det T(\xi)=4(\xi-\xi_1)(\xi-\xi_2)$ and $\det T(\xi_k)=0$ $(k=1,2)$. Since N and $\bar N$ in (2.2) and (3.2) are traceless, $\det\varphi$, $\det\bar\varphi$ do not depend on t. It follows from the relation $\det T=\det\bar\varphi/\det\varphi$ that ξ_1 and ξ_2 are independent of t and when $\xi=\xi_k$ $(k=1,2)$, $\bar\varphi$ in (3.1) are collinear, i.e.

$$\begin{pmatrix}\bar\varphi_{11}(\xi_j)\\ \bar\varphi_{21}(\xi_j)\end{pmatrix}+k_j\begin{pmatrix}\bar\varphi_{12}(\xi_j)\\ \bar\varphi_{22}(\xi_j)\end{pmatrix}=0 \qquad j=1,2 \tag{3.4}$$

(where k_j are independent of t). It can be seen that

$$\begin{aligned} &(-2\mathrm{i}\xi_1+a)\phi_1+b\phi_2=0 \qquad c\phi_1+(2\mathrm{i}\xi_1+d)\phi_2=0 \\ &(-2\mathrm{i}\xi_2+a)\psi_1+b\psi_2=0 \qquad c\psi_1+(2\mathrm{i}\xi_2+d)\psi_2=0 \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} &\phi_1=\varphi_{11}(\xi_1)+k_1\varphi_{12}(\xi_1) \qquad \psi_1=\varphi_{11}(\xi_2)+k_2\varphi_{12}(\xi_2) \\ &\phi_2=\varphi_{21}(\xi_1)+k_1\varphi_{22}(\xi_1) \qquad \psi_2=\varphi_{21}(\xi_2)+k_2\varphi_{22}(\xi_2). \end{aligned} \tag{3.6}$$

By solving equation (3.5), we obtain

$$\begin{aligned} &a=(2\mathrm{i}\xi_1\phi_1\psi_2-2\mathrm{i}\xi_2\phi_2\psi_1)/\Delta \qquad d=(2\mathrm{i}\xi_1\phi_2\psi_1-2\mathrm{i}\xi_2\phi_1\psi_2) \\ &b=2\mathrm{i}(\xi_2-\xi_1)\phi_1\psi_1/\Delta \qquad c=2\mathrm{i}(\xi_2-\xi_1)\phi_2\psi_2/\Delta \end{aligned} \tag{3.7}$$

$$\Delta=\phi_1\psi_2-\phi_2\psi_1. \tag{3.8}$$

Substituting b and c defined by (3.7) into (3.3), we obtain

$$\bar{q} = -q + 2\mathrm{i}(\xi_2 - \xi_1)\phi_1\psi_1/\Delta \qquad \bar{r} = -r + 2\mathrm{i}(\xi_2 - \xi_1)\phi_2\psi_2/\Delta. \tag{3.9}$$

We call (3.1) and (3.9) the Darboux transformation of (2.2). It means that the new $\bar{q}$, $\bar{r}$ and $\bar{\varphi}$ can be expressed by the old q and r and their eigenfunctions φ. Next we shall prove the following theorem.

Theorem. If q, r is the solution of (2.5), and φ is a fundamental solution of (2.1) and (2.2), then $\bar{q}$, $\bar{r}$ defined by (3.9) is another solution of (2.5).

Proof. Let

$$\bar{M} = (\bar{D}T + TM)T^{-1}. \tag{3.10}$$

From lemma 5 of Li and Gu (1987) M and $\bar{M}$ are both found to be quadratic polynomials of ξ:

$$\bar{M} = \sum_{j=0}^{2} \begin{pmatrix} a_j & b_j \\ c_j & -a_j \end{pmatrix} \xi^j. \tag{3.11}$$

We rewrite (3.10) as $\bar{D}T = \bar{M}T - TM$ and substitute M, $\bar{M}$, T, defined by (2.1), (3.11) and (3.1) respectively, into it. We have

$$\begin{aligned}
\bar{D}\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= 2\mathrm{i}\xi^3 \begin{pmatrix} -\bar{a}_2 + a_2 & \bar{b}_2 \\ -\bar{c}_2 & -\bar{a}_2 - a_2 \end{pmatrix} \\
&\quad + \begin{pmatrix} 2\mathrm{i}(\bar{a}_1 - a_1) + (\bar{a}_2 - a_2)a & 2\mathrm{i}(\bar{b}_1 + b_1) + (\bar{a}_2 + a_2)b \\ -2\mathrm{i}(\bar{c}_1 + c_1) - (\bar{a}_2 - a_2)c & 2\mathrm{i}(\bar{a}_1 - a_1) + (\bar{a}_2 - a_2)d \end{pmatrix} \\
&\quad + \xi \begin{pmatrix} 2\mathrm{i}(-\bar{a}_0 + a_0) + (\bar{a}_1 a + \bar{b}_1 c) - (aa_1 + bc_1) & 2\mathrm{i}(\bar{b}_0 + b_0) + (\bar{a}_1 b + \bar{b}_1 d) - (ab_1 - ba_1) \\ 2\mathrm{i}(-\bar{c}_0 - c_0) + (\bar{c}_1 a + \bar{a}_1 c) - (ca_1 + dc_1) & 2\mathrm{i}(-\bar{a}_0 + a_0) + (\bar{c}_1 b - \bar{a}_1 d) - (cb_1 - da_1) \end{pmatrix} \\
&\quad + \begin{pmatrix} \bar{a}_0 a + \bar{b}_0 c - (ab_0 + bc_0) & \bar{a}_0 b + \bar{b}_0 d - (ab_0 - ba_0) \\ \bar{c}_0 a - \bar{a}_0 c - (ca_0 + dc_0) & \bar{c}_0 b - \bar{a}_0 d - (cb_0 - da_0) \end{pmatrix}.
\end{aligned}$$

Equating the coefficient of ξ^j, $j = 3, 2, 1$, respectively, we obtain

$$\begin{aligned}
&\bar{a}_1 = -2\mathrm{i} && \bar{b}_2 = \bar{c}_2 = 0 \\
&\bar{a}_1 = a = \mathrm{i}\alpha && \bar{b}_1 = 2\bar{q} \qquad \bar{c}_1 = 2\bar{r} && \text{(using (3.3}a\text{))} \\
&\bar{a}_0 = -\mathrm{i}qr && \bar{b}_0 = \bar{q}_t + \alpha\bar{q} \qquad \bar{c}_0 = -\bar{r}_t + \alpha\bar{r} && \text{(using (3.3}a\text{), (3.3}c\text{))}.
\end{aligned}$$

It turns out that $\bar{M}$ is the same type as M but q, r, q_t, r_t are interchanged with $\bar{q}$, $\bar{r}$, $\bar{q}_t$, $\bar{r}_t$ which are defined by (3.9) and $\bar{D}\bar{\varphi} = \bar{M}\bar{\varphi}$. Since in (3.2) $\bar{N}$ is the same type as N but q, r are interchanged with $\bar{q}$, $\bar{r}$, from the compatibility condition $\partial_t\bar{D}\bar{\varphi} = \bar{D}\partial_t\bar{\varphi}$, $\bar{M}_t - \bar{D}\bar{N} + \bar{M}\bar{N} - \bar{N}\bar{M} = 0$. It means that $\bar{q}$, $\bar{r}$ defined by (3.9) also satisfies (2.5).

Remark. Since $\bar{M}$, M are both traceless, ξ_1, ξ_2, k_1 and k_2 in (3.5)–(3.7) are constants (independent of x, y and t).

4. Some special solutions of equation (2.6)

The important case occurs when $q=-r^*=S$, $\bar{q}=-\bar{r}^*=\bar{S}$. In this case, $\xi_2=\xi_1^*$, $\psi_1=\phi_2^*$, $\psi_2=-\phi_1^*$, and (3.9) is reduced as follows:

$$\bar{S}=-S-\frac{2i(\xi_1^*-\xi_1)\phi_1\phi_2^*}{\phi_1\phi_1^*+\phi_2\phi_2^*}. \tag{4.1}$$

(i) Since $S=0$ is a solution of (2.6), we take $S=0$ as our 'seed'. The fundamental solution of the equation

$$\varphi_{1t}=-i\xi\varphi_1 \qquad \varphi_{2t}=i\xi\varphi_2 \tag{4.2}$$

$$\varphi_{1x}+\beta\varphi_{1y}=(-2i\xi^2-i\alpha)\varphi_1 \qquad \varphi_{2x}+\beta\varphi_{2y}=(2i\xi^2+i\alpha)\varphi_2 \tag{4.3}$$

is

$$\varphi=\begin{pmatrix}\varphi_{11} & \varphi_{12}\\ \varphi_{21} & \varphi_{22}\end{pmatrix}=\begin{pmatrix}f_1(z)\exp(g(x,t,\xi)) & 0\\ 0 & f_2(z)\exp(-g(x,t,\xi))\end{pmatrix} \tag{4.4}$$

where $f_1(z)$ and $f_2(z)$ are arbitrary functions of z with $z=y-\beta x$,

$$g(x,t,\xi)=-i(\xi t+2\xi^2x+\alpha x\xi). \tag{4.5}$$

We take

$$\xi_1=\zeta+i\eta \qquad \phi_1/\phi_2=f(z)\exp(2g(x,t,\xi)) \qquad f(z)=\exp(\delta(z)+i\theta(z)). \tag{4.6}$$

From (4.1), we obtain

$$\bar{S}=\frac{-2\eta\exp\{-4i[\zeta^2+\eta^2+(\alpha/2)\zeta]x-2i\zeta t+i\theta(y-\beta x)\}}{\cosh[8(\zeta+\alpha/4)\eta x+2\eta t+\delta(y-\beta x)]}. \tag{4.7}$$

When $\beta=0$, equation (2.5) is reduced to $(1+1)$ dimensions. The function S defined by (4.7) is a function of x, t, $\delta(y)$ and $\theta(y)$ are constants. This is just a solution obtained by the inverse scattering method (Ma 1984). We put $\zeta=-\alpha/4$, $\delta=0$ in (4.7); S is reduced to

$$S=-2\eta\exp[4i(\alpha^2/16+\eta^2)x+\tfrac{1}{2}i\alpha t+i\theta]\}\cosh 2\eta t. \tag{4.8}$$

This is a periodic solution of (2.6)—it does not decay to zero as $|x|\to\infty$. The inverse scattering method presented in Ma (1984) cannot be applied, but it is not difficult to deduce by the Darboux transformation.

To continue this procedure, we take $\bar{S}$ defined by (4.8) with $\theta=0$, as our 'seed'; from (3.7), we have

$$\begin{aligned}
a&=-\tfrac{1}{2}i\alpha+2\eta\tanh 2\eta t\\
d&=\tfrac{1}{2}i\alpha-2\eta\tanh 2\eta t\\
b&=-2\eta\exp\{i[\tfrac{1}{2}\alpha t+(4\eta^2+\tfrac{1}{4}\alpha^2)x]\}\cosh 2\eta t\\
c&=2\eta\exp\{-i[\tfrac{1}{2}\alpha t+(4\eta^2+\tfrac{1}{4}\alpha^2)x]\}\cosh 2\eta t.
\end{aligned} \tag{4.9}$$

From (3.1) and (4.4), we obtain

$$\begin{aligned}
\bar{\varphi}_{11}&=(-2i\xi+a)\,e^{g} & \bar{\varphi}_{12}&=b\,e^{-g}\\
\bar{\varphi}_{21}&=c\,e^{g} & \bar{\varphi}_{22}&=(2i+d)\,e^{-g}.
\end{aligned} \tag{4.10}$$

From (3.6), we take $\bar{k}_1=1$ and another special value $\bar{\xi}_1=\mathrm{i}\varepsilon$, $\bar{\xi}_2=\bar{\xi}_1^*$ and

$$\begin{aligned}\bar{\phi}_1&=\bar{\varphi}_{11}(\mathrm{i}\varepsilon)+\bar{\varphi}_{12}(\mathrm{i}\varepsilon) \qquad \bar{\psi}_1=\bar{\phi}_2^*\\ \bar{\phi}_2&=\bar{\varphi}_{21}(\mathrm{i}\varepsilon)+\bar{\varphi}_{22}(\mathrm{i}\varepsilon) \qquad \bar{\psi}_2=-\bar{\phi}_1^*.\end{aligned} \tag{4.11}$$

The other new solution of (2.6) is

$$\bar{\bar{S}}=-\bar{S}-\frac{2\mathrm{i}(\bar{\xi}_1^*-\bar{\xi}_1)\bar{\phi}_1\bar{\phi}_2^*}{\bar{\phi}_1\bar{\phi}_1^*+\bar{\phi}_2\bar{\phi}_2^*} \tag{4.12}$$

where

$$-\bar{S}=2\eta\exp[\mathrm{i}(\tfrac{1}{4}\alpha^2+4\eta^2)x+\tfrac{1}{2}\mathrm{i}\alpha t]\cosh 2\eta t$$

$$\bar{\phi}_1\bar{\phi}_1^*+\bar{\phi}_2\bar{\phi}_2^*=-2[(4\varepsilon^2+\tfrac{1}{4}\alpha^2+4\eta^2+8\varepsilon\eta\tanh 2\eta t)\cosh 2a+2fh\cos(p-2b) \tag{4.13}$$

$$\begin{aligned}2\mathrm{i}(\bar{\xi}_1^*-\bar{\xi}_1)\bar{\phi}_1\bar{\phi}_2^*=[&(f^2+g^2)\exp(2b\mathrm{i})+h^2\exp(2\mathrm{i}p-2\mathrm{i}b)\\ &+2h\exp(\mathrm{i}p)(f\cosh 2a+\mathrm{i}g\sinh 2a)]\end{aligned} \tag{4.14}$$

$$\begin{aligned}f&=(2\varepsilon+2\eta\cosh 2\eta t) \qquad g=-\tfrac{1}{2}\alpha\\ h&=-2\eta/\cosh 2\eta t \qquad p=\tfrac{1}{2}\alpha t+(4\eta^2+\tfrac{1}{4}\alpha^2)x.\end{aligned} \tag{4.15}$$

(ii) We take the 'seed'

$$S=A\,\mathrm{e}^{\mathrm{i}\omega t} \tag{4.16}$$

where A, ω are constants and satisfy the relation

$$2A^2=\omega^2-\alpha\omega. \tag{4.17}$$

The fundamental solution of the equation

$$\varphi_{1t}=-\mathrm{i}\xi\varphi_1+A\,\mathrm{e}^{\mathrm{i}\omega t}\varphi_2 \qquad \varphi_{2t}=-A\,\mathrm{e}^{-\mathrm{i}\omega t}\varphi_1+\mathrm{i}\xi\varphi_2 \tag{4.18}$$

$$\varphi_{1x}+\beta\varphi_{1y}=\mathrm{i}g\varphi_1+Af\,\mathrm{e}^{\mathrm{i}\omega t}\varphi_2 \qquad \varphi_{2x}+\beta\varphi_{2y}=-fA\,\mathrm{e}^{-\mathrm{i}\omega t}\varphi_1-\mathrm{i}g\varphi_2 \tag{4.19}$$

where

$$f=2\xi-\omega+\alpha \qquad g=-2\xi^2-2\alpha\xi+A^2 \tag{4.20}$$

is

$$\varphi=\begin{pmatrix}\exp\left(\frac{\mathrm{i}}{2}(\omega-\sqrt{\Delta_1})t-\frac{\mathrm{i}f}{2}\sqrt{\Delta_1}\,x\right) & \frac{A}{\mathrm{i}(\xi+\frac{1}{2}\omega+\frac{1}{2}\sqrt{\Delta_1})}\exp\left(\frac{\mathrm{i}}{2}(\omega+\sqrt{\Delta_1})t+\frac{\mathrm{i}f}{2}\sqrt{\Delta_1}\,x\right)\\ \frac{A}{\mathrm{i}(\xi+\frac{1}{2}\omega+\frac{1}{2}\sqrt{\Delta_1})}\exp\left(\frac{\mathrm{i}}{2}(-\omega-\sqrt{\Delta_1})t-\frac{\mathrm{i}f}{2}\sqrt{\Delta_1}\,x\right) & \exp\left(\frac{\mathrm{i}}{2}(-\omega+\sqrt{\Delta_1})t+\frac{\mathrm{i}f}{2}\sqrt{\Delta_1}\,x\right)\end{pmatrix} \tag{4.21}$$

where

$$\Delta_1=4\xi^2+4\omega\xi+\xi\omega^2-2\alpha\omega=4A^2+\omega^2+4\omega\xi+4\xi^2. \tag{4.22}$$

(a) When $\omega=A=0$, it reduces to (i).

(b) $\Delta_1=0$, $\xi_1=-\frac{1}{2}\omega-\mathrm{i}A$, $\phi_1=(1+k_1)\exp(\mathrm{i}\omega t/2)$, $\phi_2=(1+k_1)\exp(-\mathrm{i}\omega t/2)$. Substituting ξ_1, ϕ_1, ϕ_2 into (4.1), we obtain

$$\bar{S}=A\,\mathrm{e}^{\mathrm{i}\omega t}. \tag{4.23}$$

This is one interesting example which says that under the Darboux transformation the solution is invariant.

$$\text{(c)}\ \Delta_1=\omega \qquad \omega>2\alpha \qquad \xi_1=-\frac{\omega}{2}+\mathrm{i}(\omega^2-2\alpha\omega)^{1/2} \tag{4.24}$$

$$\mathrm{i}\left(\xi+\frac{\omega}{2}+\frac{\sqrt{\Delta_1}}{2}\right)=-\frac{(\omega^2-2\alpha\omega)^{1/2}}{2}+\mathrm{i}\frac{\omega}{2}=A\,\mathrm{e}^{\mathrm{i}\theta_1} \qquad \theta_1=\tan^{-1}\frac{\omega}{(\omega^2-2\alpha\omega)^{1/2}} \tag{4.25}$$

$$-\frac{\mathrm{i}f\omega}{2}=\frac{\omega}{2}[(\omega^2+2\alpha\omega)^{1/2}+\mathrm{i}(2\omega-\alpha)]=\frac{\omega\gamma}{2}\mathrm{e}^{\mathrm{i}\theta} \qquad \theta=\tan^{-1}\frac{2\omega-\alpha}{(\omega^2-2\alpha\omega)^{1/2}}$$

$$\gamma^2=5\omega^2-6\omega\alpha+\alpha^2. \tag{4.26}$$

In this case, relation (4.21) is

$$\varphi=\begin{pmatrix}\exp[\frac{1}{2}\omega x\gamma(\cos\theta+\mathrm{i}\sin\theta)] & \exp[\mathrm{i}\theta_1-\frac{1}{2}\omega x\gamma(\cos\theta+\mathrm{i}\sin\theta)+\mathrm{i}\omega t] \\ \exp[\mathrm{i}\theta_1+\frac{1}{2}\omega x\gamma(\cos\theta+\mathrm{i}\sin\theta)-\mathrm{i}\omega t] & \exp[-\frac{1}{2}\omega x\gamma(\cos\theta+\mathrm{i}\sin\theta)]\end{pmatrix}.$$

We take

$$\phi_1=R\exp\left(\mathrm{i}\varphi+\frac{\omega}{2}\gamma(\cos\theta+\mathrm{i}\sin\theta)x\right)+R\exp\left(-\mathrm{i}\varphi-\mathrm{i}\theta+\mathrm{i}\omega t-\frac{\omega}{2}\gamma(\cos\theta+\mathrm{i}\sin\theta)x\right)$$

$$\phi_2=R\exp\left(\mathrm{i}\varphi-\mathrm{i}\theta_1-\mathrm{i}\omega t+\frac{\omega}{2}\gamma(\cos\theta+\mathrm{i}\sin\theta)x\right)$$
$$+R\exp\left(-\mathrm{i}\varphi-\frac{\omega}{2}\gamma(\cos\theta+\mathrm{i}\sin\theta)x\right). \tag{4.27}$$

From (4.1), we obtain

$$S=-A\,\mathrm{e}^{\mathrm{i}\omega t}\frac{2(\omega^2-\omega\alpha)^{1/2}\,\mathrm{e}^{\mathrm{i}\omega t}[\cos(\omega\gamma\sin\theta x-\omega t+2\varphi)+\cos\theta_1\cosh\omega\gamma\cos\theta x+\mathrm{i}\sin\theta_1\sinh\omega\gamma\cos\theta x]}{[\cosh\omega\gamma\cos\theta x-\cos\theta_1\cos(\omega\gamma-\cos\theta x-\omega t+2\varphi)]}. \tag{4.28}$$

(iii) We take $S=\exp[\mathrm{i}(ax+by-t)]$, as our 'seed', where $a=1-b\beta-\alpha$, and S is a periodic solution of (2.6).

The fundamental solution of the equation

$$\varphi_{1t}=-\mathrm{i}\xi\varphi_1+S\varphi_2 \qquad \varphi_{2t}=-S^*\varphi_1+\mathrm{i}\xi\varphi_2 \tag{4.29}$$

$$\varphi_{1x}+\beta\varphi_{1y}=\mathrm{i}g\varphi_1+fS\varphi_2 \qquad \varphi_{2x}+\beta\varphi_{2y}=-fS^*\varphi_1-\mathrm{i}g\varphi_2 \tag{4.30}$$

where

$$g=-2\xi^2-\alpha\xi+1 \qquad f=2\xi+1+\alpha \tag{4.31}$$

is

$$\varphi=\begin{pmatrix}I_1(z)\exp\left(\frac{\mathrm{i}}{2}(1-\alpha+f\sqrt{\Delta_1})x+\frac{\mathrm{i}}{2}(-1+\sqrt{\Delta_1})t\right) & I_2(z)\exp\left(\frac{\mathrm{i}}{2}(1-\alpha-f\sqrt{\Delta_1})x+\frac{\mathrm{i}}{2}(-1-\sqrt{\Delta_1})t\right) \\ I_1(z)\frac{\mathrm{i}}{2}(2\xi-1+\sqrt{\Delta_1})\exp\left(\frac{\mathrm{i}}{2}(-1+\alpha+f\sqrt{\Delta_1})x+\frac{\mathrm{i}}{2}(1+\sqrt{\Delta_1})t\right) & I_2(z)\frac{\mathrm{i}}{2}(2\xi-1-\sqrt{\Delta_1})\exp\left(\frac{\mathrm{i}}{2}(-1+\alpha-f\sqrt{\Delta_1})x+\frac{\mathrm{i}}{2}(1-\sqrt{\Delta_1})t\right)\end{pmatrix} \tag{4.32}$$

$$\Delta_1=-4\xi^2+4\xi-3$$

where $I_1(z)$, $I_2(z)$ are arbitrary functions of z with $z=y-\beta x$. For simplicity, we take $\xi_1=(1+\sqrt{3}\mathrm{i})/2$ (i.e. $\Delta_1=1$) and put

$$\begin{aligned}\phi_1&=I(z)\exp[\mathrm{i}(\xi_1+1)x+\mathrm{i}t]+I^*(z)(-\mathrm{i}\xi_1)\exp[-\mathrm{i}(\xi_1+\alpha)x-\mathrm{i}t]\\ \phi_2&=I(z)(\mathrm{i}\xi_1)\exp[\mathrm{i}(\xi_1+\alpha)x+\mathrm{i}t]-I^*(z)\exp[-\mathrm{i}(\xi_1+1)x]\end{aligned}\qquad(4.33)$$

where $I(z)=R(z)\,\mathrm{e}^{\mathrm{i}\theta(z)}$ with $z=y-\beta x$. From (4.1), we obtain

$$\bar{S}=\exp[\mathrm{i}(x-\alpha x-t)]\times\left(-\exp(\mathrm{i}bz)-\sqrt{3}\frac{\mathrm{i}\sinh\sqrt{3}x+\sqrt{3}\cosh\sqrt{3}x-2\cos(2x+\alpha x+t+2\theta(z))}{2\cosh\sqrt{3}x+\sqrt{3}\cos(2x+\alpha x+t+2\theta(z))}\right).\qquad(4.34)$$

This is a periodic function of t.

Acknowledgments

The author would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

References

Li Yi-shen 1986a *International Workshop on Applied Differential Equations* ed Ziao Shuke and Pu Fuquen (Singapore: World Scientific) pp 76-90
—— 1986b *Acta Math. Appl. Sinica* **19** 196-200
—— 1987 *Chin. Ann. Math.* to be published
Li Yi-shen and Gu Xin-shen 1987 *Chin. Ann. Math.* to be published
Li Yi-shen and Wang Chen-qi 1985 *Kexue Tongbo* **16** 1209-10
Ma Y C 1984 *Stud. Appl. Math.* **70** 201-13
Wadati M, Sanuki H and Konno K 1975 *Prog. Theor. Phys.* **53** 419-36